On the Incompleteness of Classical Mechanics

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Abstract

Classical mechanics is often considered to be a quintessential example of a deterministic theory. I present a simple proof, using a construction mathematically analogous to that of the Pasadena game (Nover and Hájek [2004]), to show that classical mechanics is incomplete: there are uncountably many arrangements of objects in an infinite Newtonian space such that, although the system's initial condition is fully known, it is impossible to calculate the system's future trajectory because the total force exerted upon some objects is mathematically undefined. It is then shown how variations of this discrete system can be obtained which increasingly approximate a uniform mass distribution, similar to that underlying a related result, due to von Seeliger ([1895]). It is then argued that this incompleteness result, as well as that presented by the Pasadena game, has no real philosophical significance as it is a mathematical pseudoproblem shared by all models which attempt to aggregate infinitely many numerical values of a certain kind.

1 Introduction.

Classical mechanics, consisting of Newton's three laws of motion along with Newton's law of universal gravitation, is often considered to be a quintessential example of a deterministic theory. For example, in *Elbow Room*, Daniel Dennett states that 'So-called "classical" or Newtonian physics is deterministic' (Dennett [1984], pg. 151). What this means, exactly, is given a precise and clear statement by David Z. Albert as follows:

> Given a list of the positions of all the particles in the world at any particular time, and of how those positions are *changing*, at that time,

> as time flows forward, and of what *sorts* of particles they are, the universe's entire history, in every detail, from that time on, can in principle be calculated (if this theory is true) with certainty. (Albert [2000], pg. 2)

Unfortunately, this characterisation of classical mechanics is not entirely correct. In what follows, I shall provide a simple mathematical proof that classical mechanics is indeterministic because it is *incomplete*. This result may be of interest for several reasons. The first is that it involves a construction similar to that used by Nover and Hájek ([2004]) to generate a problem for decision theory. The second reason is that the simple proof, which makes a number of unrealistic assumptions, can be modified in such a way that the initial conditions which lead to the incompleteness result could be realised if the actual universe were Newtonian and infinite. And, finally, the third reason this proof may be of interest is that we can show that an earlier, related result known as 'Seeliger's Paradox' can be increasingly approximated a limit of a variant of the discrete systems considered here.

Given some physical theory T, there are three ways that T can fail to be deterministic. The first way T might fail to be deterministic is that, even if we have a complete state-description of the universe,[1] there will be more than one possible future history of the universe which is compatible with, and predicted by, the causal laws. Another way of putting the point is to say that, given a set of initial conditions, there will be more than one solution to the equations of motion for the entire universe. Norton ([2008]) gives a simple and elegant example in which this kind of failure of determinism can happen in classical mechanics. In Norton's example, if you position a ball at the apex of a dome of *precisely* the right shape (the shape of the dome is absolutely crucial) it turns out that the equation of motion for the ball has multiple solutions. One solution is for the ball to remain motionless forever (which is what you would expect), but there exist an infinite family of alternate solutions in which the ball remains motionless for some arbitrary interval of time, and then spontaneously moves, unprovoked, in an arbitrary direction down the side of the dome.

It is worth noting that this first type of failure of determinism is *compatible* with a strict, literal reading of the characterisation of determinism provided by Albert above. Note that Albert requires that 'the universe's entire history, in every

[1] I use this expression, rather than the formulation used by Albert in the quoted passage, because not all deterministic theories allow us to speak of the position, velocity, and type of particle *at a particular time*, for the entire universe. Albert's characterisation of determinism applies to classical mechanics because classical mechanics assumes absolute simultaneity. Special relativity, on the other hand, is actually a better example of a deterministic theory than classical mechanics, even though the relativity of simultaneity means that we cannot speak of the state of the universe *at a particular time*.

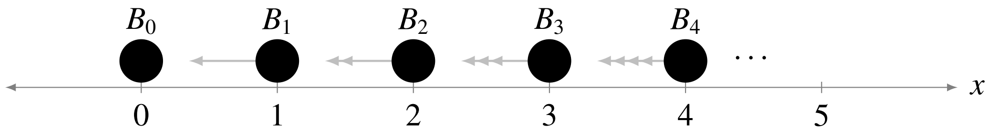


Figure 1: A configuration of balls whose collisions form a supertask, sending all of them off to infinity in finite time.

detail, from that time on, can in principle be calculated (if this theory is true) with certainty'. In the case of Norton's Dome, we *can* calculate the entire history of the universe, in every detail, from the time the universe is initially set up, with certainty. It's just that there happen to be *multiple* such histories that we can calculate with certainty.[2]

A second way that a physical theory can fail to be deterministic results from mathematical anomalies generated by the possibility of particles having unbounded velocities. Consider the following setup, due to Laraudogoitia ([1997]): in a classical system, position one ball stationary at the origin, a second ball located to its right but moving towards it at some speed, a third ball located a bit further to its right but moving towards it even faster, and so on (see figure 1). This configuration can be arranged so that the first collision between B_0 and B_1 takes place at $t = \frac{1}{2}$, the second collision between B_0 and B_1 (which happens after B_2 hits B_1) at $t = \frac{3}{4}$, and so on. It can be shown that, with a suitable selection of initial velocities, all the balls will 'disappear' to infinity in finite time. Since the laws of classical mechanics are invariant under time reversal, this means that it is possible for an infinite number of particles to spontaneously appear, 'flying in' from infinity.[3]

And the third way that a physical theory T can fail to be deterministic is that, even if we have a complete state-description of the universe, the theory T might fail to yield any answer whatsoever about the future history of the universe. This could happen if the equations of motion failed to have any solution at all. This can also happen if some of the essential quantities required by the equations of motion — quantities which, in most cases, are determined by the initial conditions — fail to be mathematically well-defined. This third kind of failure of determinism is when the physical theory T turns out to be incomplete. More precisely, a

[2]As an aside, this is an example of the rare case where a charitable interpretation of someone's *intended* meaning — rather than a strict, literal reading of what they said — actually serves to falsify their claim. I think Albert intended us to read his characterisation of determinism as implicitly including the claim that the future history of the universe, which we can calculate from its initial state, is *unique*. But this charitable interpretation transforms his characterisation of determinism from one which *isn't* falsified by the example of Norton's Dome into one which *is*.

[3]This 'space invaders' possibility was originally noted by Earman ([1986]). The fact that it is possible to achieve this, without collisions, in finite time, with a *finite* number of particles, was first proven by Xia ([1989]); a less technical discussion is provided by Saari and Xia ([1995]).

physical theory T is *incomplete* when there exist complete state descriptions of the universe, which belong to the domain of physically possible states recognised by the theory, but for which the theory does not yield any answer at all regarding the future history of the system.

Prima facie, the incompleteness of a physical theory seems interesting because it suggests that our understanding of the physical universe may be incomplete as well. If there are valid ways that particles could be arranged in the universe, and yet our current best physical theory yields no prediction as to the future state of the universe, that suggests there either may be additional physical forces of which we are unaware or that our mathematical formulation of the physical laws is not entirely correct. (At least, that is, if we assume that *something* will happen if particles are arranged in that particular way.) However, at the end of the paper I will argue that this interpretation of the incompleteness result is wrong. Instead, we should see incompleteness *of the kind identified here* as nothing more than a mathematical artefact resulting from modelling assumptions regarding infinite domains. Since it is natural, for reasons of mathematical convenience, to assume infinite domains, this means we should be cautious about taking problems generated by this kind of incompleteness as problems of philosophical significance which need to be solved.

2 The setup.

Consider the following system of particles, arranged in a line, in a classical Newtonian universe where the only force in operation is gravity. Suppose that, at the origin, we have an object with unit mass $m_0 = 1$ and at each of the points $x_k = (-1)^{k-1} \cdot k$, for $k \in \mathbb{N}$, we position an object with mass $m_k = \frac{k}{G}$, where G is the gravitational constant (see figure 2).[4] Given this initial configuration, how will the mass m_0 at the origin move?

3 The problem.

To answer this question, we need to determine the total force F of all the individual gravitational forces F_k exerted on the unit mass at the origin. By construction, the force contribution F_k of the k^{th} object is

$$F_k = G\frac{m_0 m_k}{k^2} = G\frac{1 \cdot \frac{k}{G}}{k^2} = \frac{1}{k}$$

[4]Through this paper I abuse notation quite frequently. Sometimes I shall use G as a constant without units attached, as when defining the mass m_k. Sometimes, as when calculating the gravitational force F_k exerted by the k^{th} object on m_0, I use G as the gravitational constant with units attached. Which usage is intended will be clear from the context.

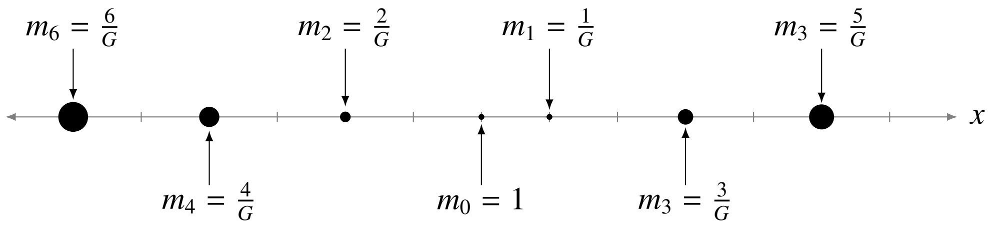


Figure 2: An illustration showing the first six particles, and their masses, located around the unit mass m_0 at the origin. Tick marks are shown at locations $x = k$ for integral values of k.

and so, if we naïvely sum all the individual forces, we see that the total force F acting on the mass m_0 is

$$F = \sum_{k=1}^{\infty} F_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \tag{1}$$

which means we have a problem.

The problem is that the series which appears in equation (1) is the alternating harmonic series, which is conditionally convergent. By the Riemann Rearrangement Theorem, the value we get for F depends on the order in which we sum the terms. If we sum the terms in the order initially noted above, we find that $F = \ln(2)$, so we would predict that the unit mass at the origin should move in the positive x direction. However, suppose we calculate F by adding the individual force contributions F_k as follows:

$$\begin{aligned} F = 1 &- \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} - \frac{1}{24} - \frac{1}{26} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} \\ &+ \frac{1}{3} - \frac{1}{34} - \frac{1}{36} - \frac{1}{38} - \frac{1}{40} - \frac{1}{42} - \frac{1}{44} - \frac{1}{46} - \frac{1}{48} - \frac{1}{50} - \frac{1}{52} - \frac{1}{54} - \frac{1}{56} - \frac{1}{58} - \frac{1}{60} - \frac{1}{62} - \frac{1}{64} \\ &+ \frac{1}{5} - \frac{1}{66} - \frac{1}{68} - \frac{1}{70} - \frac{1}{72} - \frac{1}{74} - \frac{1}{76} - \frac{1}{78} - \frac{1}{80} - \frac{1}{82} - \frac{1}{84} - \frac{1}{86} - \frac{1}{88} - \frac{1}{90} - \frac{1}{92} - \frac{1}{94} - \frac{1}{96} \\ &\vdots \end{aligned}$$

This reordering proceeds by adding sixteen 'negative' contributions for each 'positive' contribution. Under this reordering, it turns out that $F = -\ln(2)$, so we would predict that the unit mass at the origin should move in the negative x direction![5] Since classical mechanics does not specify *which* order we are to add the

[5]A rearrangement of an infinite series where a fixed number of positive terms (taken in the same order as in the original series) is followed by a fixed number of negative terms (taken in the same order as in the original series) is known as a *regular rearrangement*. Brown *et al.* ([1985]) proved that, if $A(m, n)$ denotes the sum of a regular rearrangement of the alternating harmonic series where m positive terms are following by n negative terms, then $A(m, n) = \ln(2) + \frac{1}{2}\ln\left(\frac{m}{n}\right)$.

terms, there is simply no answer to the question of how the unit mass at the origin will move: it is mathematically indeterminate. As a result, no solution exists for the Newtonian equations of motion regarding how the mass of the origin will move. This means classical mechanics is *incomplete*, as there exist initial configurations of particles in space for which the future state of the system is undefined.[6]

Those acquainted with the Pasadena game (Nover and Hájek [2004]) will recognise that what we have done is translate the underlying puzzle from its decision-theoretic context into a physical context. In the Pasadena game, a fair coin is flipped until a heads occurs, with the amount 'won' defined to be $(-1)^{k-1} \cdot \frac{2^k}{k}$, if the first head occurs on the k^{th} toss. (A negative amount means that the person playing the game actually *loses* that amount of money.) The problem raised by the Pasadena game for decision theory is that, when we go to calculate the expected value of the game, the probability-weighted payoffs yield the terms of the alternating harmonic series. Since there is no specified order in which to sum the probability-weighted payoffs, the game has no well-defined expected value.[7] Here, what we have done is arrange masses of an appropriate size, in space, so that the force contribution of the k^{th} mass corresponds to probability-weighted payoffs of the Pasadena game. If the force contribution of the k^{th} mass corresponds to a 'win' in the Pasadena game, it pulls the mass m_0 to the right (i.e., the positive direction). If the force contribution of the k^{th} mass corresponds to a 'loss' in the Pasadena game, it pulls the mass m_0 to the left (i.e., the negative direction).[8]

In the case shown, where 16 negative terms follow 1 positive term, we have

$$\begin{aligned} A(1,16) &= \ln(2) + \frac{1}{2}\ln\left(\frac{1}{16}\right) \\ &= \ln(2) + \frac{1}{2}\ln(2^{-4}) = -\ln(2). \end{aligned}$$

[6]I use the term 'incomplete' rather than 'indeterminate' because the structure of this problem differs from other examples illustrating indeterminacy in classical mechanics. In the situation of Norton's Dome (Norton; Norton [2003; 2008]), all of the relevant forces are *known*. The problem Norton identified is that *multiple* solutions existed, regarding the future trajectory for the mass at the apex of the sphere, and so the future state of the universe was not *unique*. In the case of Earman's 'space invaders' (Earman [1986]), the fact that classical mechanics places no limit on the velocity of particles means that objects can escape to infinity in finite time. The fact that the laws of classical mechanics are invariant under time reversal thus means it is possible for particles to appear from nowhere by 'zooming in' from infinity in finite time: a case of indeterminism because particles appear out of nothing in an uncaused fashion. However, both of these examples differ from this construction. Here, the total force acting on the mass at the origin depends on the order in which the individual contributing forces are summed and, since there is no answer to the question of what order we are to sum the contributing forces, the relevant force is *undefined*.

[7]Some, though, have sought to extend the valuation methods of traditional decision theory so as to provide a well-defined value of the Pasadena game (see Easwaran [2008] for details).

[8]Essentially, this is the gravitational analogue of the puzzle considered by Linnebo ([2023]).

4 Old wine in a new bottle?

Cognoscenti will recognise that the problem described above is similar in spirit to a problem first pointed out by Hugo Seeliger in 1895. 'Seeliger's Paradox', as it became known, is that when a unit mass in an infinite universe is under the influence of a uniform mass distribution, then the gravitational force exerted on the unit mass fails to converge. (For a discussion of the history of Seeliger's paradox, and the various solutions proposed, see Norton [1999].)

Although it might appear that we have here nothing more than old wine in a new bottle, there are three reasons why the above construction is of interest. The first reason is that whereas Seeliger's proof required a continuous uniform mass distribution throughout an infinite universe, the example provided here is *discrete*. Furthermore, since the key fact needed to obtain the incompleteness result is that the alternating harmonic series is conditionally convergent, we can in principle choose the masses m_k appropriately so as to generate F_k corresponding to any conditionally convergent sequence. This shows that the problem isn't just limited to a continuous uniform mass distribution. If we assume that it is possible to postulate point masses of arbitrary size, then there are actually uncountably many different initial configurations for which the gravitational force fails to converge.

The second reason is that, as we will see in the next section, it is possible to obtain a formulation of the problem which would be, at least in principle, realisable if the actual universe were Newtonian. This is somewhat surprising, since both Seeliger's original paper and the presentation of problem in section 2 involve some unrealistic assumptions. It turns out that these unrealistic assumptions actually play no significant role in establishing the incompleteness result.

The third reason that this statement of the problem might be of interest is that, as we will see in section 6, that by taking a suitable limit of our problem we are able to generate a sequence of discrete incompleteness results that approximates, with ever-greater accuracy, the uniform mass distribution underlying Seeliger's paradox. Let us now examine these last two points in detail.

5 Keep it real.

There are a couple of concerns which might be raised regarding the construction given in section 2. The first is that the numerical value of the gravitational constant G is quite small — $6.674 \times 10^{-11} \frac{N \cdot m}{kg^2}$ — which means that the mass of the first object, $\frac{1}{G}$, is quite large: slightly more than 14,983,518,130 kilograms. The second is that the mass of the k^{th} object increases without bound. If one purported

For a detailed analysis of that problem, with a focus on its metaphysical implications, see Andrew Lee's 'A Puzzle about Sums', forthcoming in *Oxford Studies in Metaphysics*.

'benefit' of the construction given is that it is discrete, it is unclear whether this really counts as an improvement. Perhaps all we have shown is that classical mechanics cannot completely describe a universe in which point objects with unbounded mass can exist. Malament ([2008]) observes that all kinds of trouble can be generated for classical mechanics if we posit (i.e., 'make up') *forces* without any restriction. Perhaps what we've seen just shows that further trouble can be generated if we posit *objects* without any restriction.

For this reason, I now show that it is possible to obtain a slightly more realistic formulation of the problem, one which requires nothing more than ordinary particles positioned in a Newtonian universe.[9] To see how, we proceed in stages. We will first show how the need to posit objects of unbounded mass can be avoided, constructing a version of the incompleteness result which requires nothing more than objects with a mass of $\frac{1}{G}$. If we then replace the objects with a mass of $\frac{1}{G}$ with, say, neutrons, we obtain a version of the incompleteness result using known particles, based on a rescaled version of the alternating harmonic series.

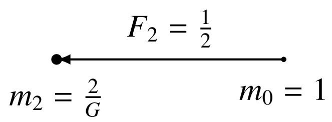
We thus need to show how, given an object m_k of mass $\frac{k}{G}$ which exerts a gravitational force F_k upon the object m_0, it is possible to construct an arrangement of some (finite) number of objects, each having a mass of $\frac{1}{G}$, such that the aggregate force of that finite set of objects upon m_0 is equivalent to F_k. Let us call this a *gravitationally equivalent decomposition* of the force F_k. That is the first task. The second task is to show that this equivalent construction can be done for *each* object posited in section 2 in a way that doesn't require the set of objects needed for each decomposition to overlap, or become arbitrary close, in space.

To begin, note that there is nothing to do for the object m_1, since that is a single object with mass $\frac{1}{G}$ already.[10] Now consider the object m_2, with a mass of $\frac{2}{G}$, positioned at $x = -2$, which exerts a force of $F_2 = \frac{1}{2}$ on m_0. Notice that if we symmetrically position *two* objects with mass $\frac{1}{G}$ at just the right height above and below the x-axis, as shown in 3(b), the vertical forces exerted on m_0 will cancel

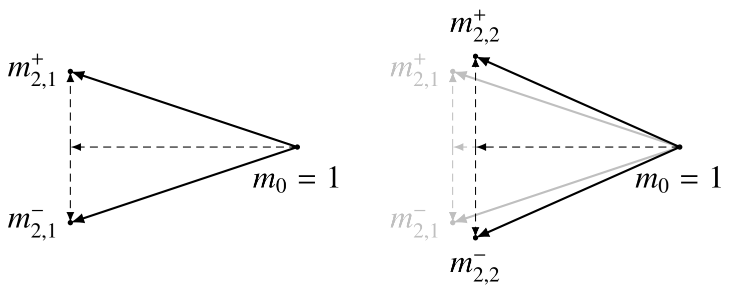
[9]It is important, though, not to overstate the degree to which this version of the problem is 'slightly more realistic'. We will show that the incompleteness result holds for ordinary particles, such as neutrons, located at the appropriate points in space. The way to think of this is as an initial configuration of the universe, possibly created by God, since the configuration falls within the domain of possible states according to Newtonian physics. I make no claim as to whether Newtonian physics could hold during the creation or formation of this state. I suspect not, because if we cannot calculate *future* states of the universe, for some configuration C, because key quantities are mathematically undefined, it is unclear if we could calculate *past* states of the universe which led to the configuration C. Since the laws of classical mechanics are symmetric with respect to time, I suspect this would not be possible.

[10]That said, note that the reasoning involved in this paragraph could also be applied to m_1: it is possible to position two objects, each with mass $\frac{1}{G}$, at $x = 1$ with some vertical displacement d, such that the total gravitational force exerted on m_0 is equivalent to that of m_1. This is shown in later figures, and demonstrated in Appendix A.

and the sum of the horizontal forces from the two symmetrically placed objects exerted on m_0 will be equivalent to that of a *single* object with mass $\frac{1}{G}$ placed at $x = -2$, namely a force of $\frac{1}{4}$. (Calculations for this and remaining diagrams can be found in appendix A.) Now, suppose we symmetrically place two additional masses some distance ϵ *closer* to m_0 at the same vertical displacement as before. Since these two additional masses are now closer to m_0, although the vertical forces exerted on m_0 will cancel, the sum of the horizontal forces will now be *greater* than a single object with mass $\frac{1}{G}$ placed at $x = -2$. In order to ensure that the sum of the horizontal forces is what we want, the vertical displacement of these two additional masses will need to be slightly greater. The final arrangement is shown in figure 3(c). By construction, the net gravitational force of the four objects with mass $\frac{1}{G}$ at the various positions shown is equivalent to the gravitational force of a single object with mass $\frac{2}{G}$ located at $x = -2$.



(a) The target



(b) Step 1 (c) Step 2

Figure 3: How to construct a gravitationally equivalent decomposition of m_2.

This construction can be repeated. For the case of the object m_3, located at $x = 3$ with a mass of $\frac{3}{G}$, we need to position six objects, each having a mass of $\frac{1}{G}$ at suitable positions along the x-axis so that all of the vertical force components exerted on m_0 cancel, and all the horizontal components sum to $F_3 = \frac{1}{3}$. This completes the proof-sketch of the first task: that it is always possible to perform a gravitationally equivalent decomposition of the object m_N, with mass $\frac{N}{G}$, into $2N$ objects, each with a mass of $\frac{1}{G}$, such that the total force exerted on m_0 is the same.

Now let us turn to the second task: to show that the gravitationally equivalent decomposition can be done without requiring the set of objects to overlap, or to become arbitrarily close in space. To see this, notice that the decomposition of

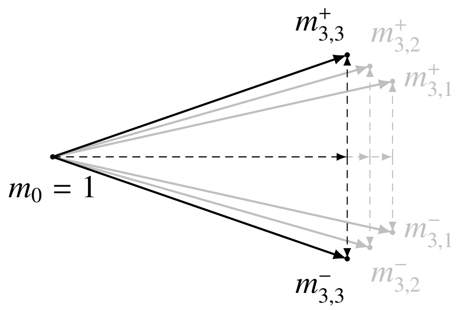


Figure 4: How to construct a gravitationally equivalent decomposition of m_3.

object m_k involves positioning k *pairs* of objects, with the first pair located at $x_k = k$ (if k is odd) or at $x_k = -k$ (if k is even), and then moving ϵ closer towards the origin, with each pair of objects generating a horizontal-force equivalent of $\frac{1}{k^2}$ on m_0.[11] As can be seen from the calculations in the appendix, not only is each pair of objects no closer than ϵ away (horizontally) from any other object, but the coordinate calculated for $y_{2(k+1),1}$ is greater than the coordinate calculated for $y_{2k,2k}$. This means that the objects which appear in the decomposition will be isolated both horizontally and vertically by a distance of at least ϵ.

Now consider the following situation: suppose that we have an object of unit mass positioned at the origin and at the points $x_k = (-1)^{k-1} \cdot k$, for $k \in \mathbb{N}$, we imagine positioning an object with the mass of k neutrons. Given that problem, determine the gravitationally equivalent decomposition with pairs of neutrons. In the decomposition, every neutron will be at least ϵ away from every other neutron, and since we can safely take $\epsilon = \frac{1}{10}$ each neutron will be isolated in space. In this new problem, the force contribution of the k^{th} decomposed set will be

$$F_k = G\frac{m_0 \cdot (kn^o)}{k^2} = G\frac{n^o}{k}$$

where n^o denotes the mass of a neutron. Attempting to naïvely sum all the forces in *this* problem yields

$$\begin{aligned} F = \sum_{k=1}^{\infty} F_k &= G\frac{n^o}{1} - G\frac{n^o}{2} + G\frac{n^o}{3} - G\frac{n^o}{4} + G\frac{n^o}{5} - G\frac{n^o}{6} + \cdots \\ &= Gn^o\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right). \end{aligned}$$

This new series is nothing more than the alternating harmonic series rescaled by a very small positive constant. The series is still conditionally convergent, and

[11] When each pair of objects generates a horizontal-force equivalent of $\frac{1}{k^2}$, and we have k pairs the aggregate force on m_0 is thus $\frac{k}{k^2} = \frac{1}{k}$, which is the target value F_k.

hence classical mechanics yields no answer to the question of how the unit mass at the origin will move.

6 Taking the limit.

There are two further points worth noting. The first is that, as shown in figure 5, the distribution of mass required for this incompleteness result is quite sparse when compared with Seeliger's paradox. Instead of a continuous, uniform distribution of mass throughout the universe, this discrete version of the problem, featuring point particles, involves a virtually empty universe with measure zero of the space occupied by mass.

The second, somewhat surprising, point is that, if we consider variant formulations of the problem, we find that our Pasadena-game inspired situation allows us to obtain ever-greater approximations of Seeliger's paradox in the limit. To see how, first consider what happens if we construct the gravitationally equivalent decomposition using objects with a mass of $\frac{1}{10G}$. In this case, we will now need 10 pairs of objects to replace m_1, 20 pairs of objects to replace m_2, and so on. If we keep $\epsilon = \frac{1}{10}$, then the pairs of objects introduced in the decomposition of m_k will actually span the interval from $x = (-1)^{k-1} \cdot k$ to within ϵ of $x = 0$, in steps of ϵ. Here, as shown in figure 6(a), the pairs of objects form graceful arcs, converging towards the origin.

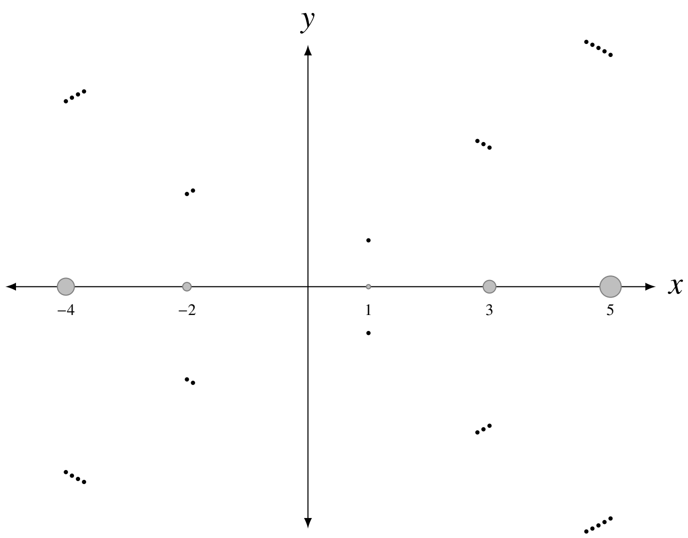
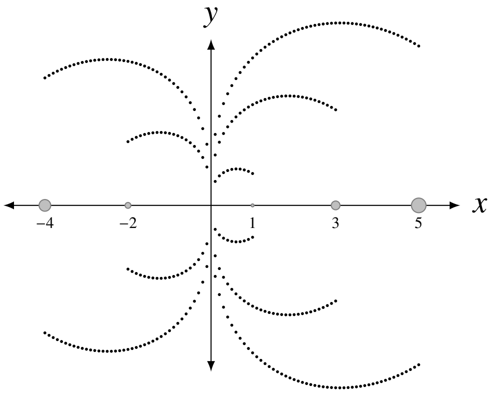
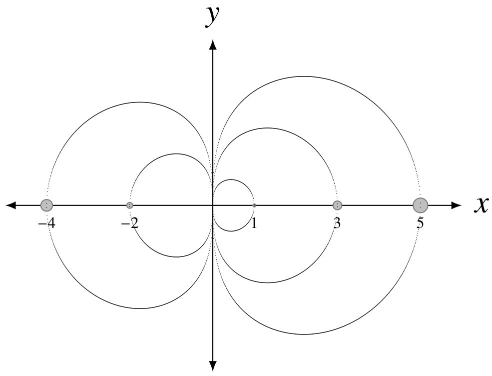


Figure 5: The distribution of particles in space for the gravitationally equivalent decomposition of the first four masses. The target mass is shown as a gray disk.



(a) The distribution of particles in space for a gravitationally equivalent decomposition using pairs of objects having an *individual* mass of $\frac{1}{10G}$ instead of $\frac{1}{G}$.



(b) The resulting distribution of particles in space for a gravitationally equivalent decomposition of the original problem but using pairs of objects having a *total* mass of slightly more than $\frac{1}{100G}$.

Figure 6: Two variant gravitationally equivalent decompositions which increasingly approximate smooth arcs towards the origin.

In fact, if we construct the gravitationally equivalent decomposition using even smaller pairs of masses of the right value, the pairs will form ever-greater approximations to continuous arcs approaching the origin. Figure 6(b) illustrates this for the case where each object in the pair has a mass just slightly exceeding $\frac{1}{200G}$, where 100 pairs are used to replace m_1, 200 pairs to replace m_2, and so on. The important takeaway message from this figure is that it suggests that it may be possible to increasingly approximate Seeliger's paradox, in the limit, if we consider suitably rescaled versions of the problem, starting with a case where the initial objects are positioned at $x_k = (-1)^{k-1} \cdot \frac{k}{10^n}$, for some fixed n and $k \in \mathbb{N}$. (Details of the construction can be found in Appendices A and B.)

7 So what?

Here is a brief summary of what we have shown: by translating the basic problem of the Pasadena game into a physical context, we have obtained a new proof that classical mechanics is incomplete. Furthermore, the incompleteness of classical mechanics does not rest on particularly unrealistic assumptions, such as peculiar forces or objects of unbounded mass. And, finally, by performing a gravitationally equivalent decomposition in the right way, it can be shown that our discrete formulation increasingly approximates Seeliger's paradox.

What's the overarching philosophical significance? There is a shallow lesson to be drawn and, I think, a deeper one. The shallow lesson is simply to correct bad philosophical practice: although some people cite classical mechanics as a canonical example of a deterministic physical theory, it really isn't. The shallow lesson, then, continues the project started by Earman ([1986]), in trying to correct misimpressions that are wide-spread throughout the philosophical literature. As Earman said, 'when told that classical physics is not the place to look for clean and unproblematic examples of determinism, most philosophers react with a mixture of disbelief and incomprehension.' One benefit of translating the Pasadena game into the context of classical mechanics is that it gives a relatively simple illustration of how classical mechanics fails to be deterministic by being incomplete, a point which has been insufficiently appreciated.

It seems to me that we can distinguish two different senses in which classical mechanics is incomplete.[12] I don't think these two senses are mutually exclusive, and both might prove to be philosophically interesting notions to explore. The first sense is the one I have stressed in this paper: that there exist complete state-descriptions of physical systems — where all of the initial positions, velocities and masses of particles are known, along with all of the individual pairwise forces exerted between particles — and yet classical mechanics fails to provide an answer to the question of how the state of the system will evolve in the future. This first sense is an incompleteness of the solutions of the equations of motion: some physical configurations simply do not have any solutions whatsoever.

However, we could adopt an alternative diagnosis of what has gone wrong. The alternative diagnosis would locate the incompleteness of classical mechanics in its failure to specify precisely *which* complete state-descriptions of physical systems are actually physically possible states appearing in the domain of the theory. This second sense of incompleteness, then, relates to the statement given by Albert at the start of this paper:

> Given a list of the positions of all the particles in the world at any particular time, and of how those positions are *changing*, at that time, as time flows forward, and of what *sorts* of particles they are, the universe's entire history, in every detail, from that time on, can in principle be calculated (if this theory is true) with certainty. (Albert [2000], pg. 2)

Classical mechanics is incomplete because, clearly, not *every* such list allows us to calculate the future history of the universe with certainty. We need more information as to *which* such lists are physically valid (or viable) initial states, recognised by the theory. Until we have that information, classical mechanics is incomplete.

[12] I would like to thank an anonymous referee for suggesting that I consider including a short discussion along these lines.

The deeper lesson requires us to ask why, or whether, we should take the incompleteness result seriously in the first place. At the outset, I suggested that incompleteness results for physical theories were *prima facie* interesting because they suggested the possibility that we didn't fully understand the physics of the world. If the world *could be* a certain way (that is, if the postulated initial conditions are in the domain of the theory), and we expect something to *happen*, if the world were that way, then clearly if we cannot say what would happen, if the world were that way, then there is something we don't know. But I suggest it would be a mistake to infer that the incompleteness result says something interesting about what we don't know *regarding physical theory*. It could be the case that our knowledge of physical theory is basically correct, but there are additional metaphysical constraints, of which we are unaware, which exclude those problematic states for which the physical theory yields no prediction. As, for example, would happen if the universe were finite but just really, really large. If this were the case, the apparent incompleteness result would simply be a pseudoproblem resulting from unwarranted reification of mathematical assumptions (e.g., an infinite universe) introduced for reasons of simplicity. This point resonates with an argument made by Wilson ([2009]).[13]

Consider the incompleteness result discussed in this paper. In addition to the laws of classical mechanics, it assumes an infinite universe in which infinitely many particles (of some kind) can be positioned arbitrarily. It assumes, in other words, an infinite dimensional state space where every point in that state space falls within the domain of the theory. There are a number of reasons why it might be mathematically convenient to model the physical universe that way, but it is a mistake to think that mathematically convenient modelling assumptions should yield insight regarding what is physically possible.[14] We use mathematics to de-

[13]In that paper, Wilson argued that many apparent cases of indeterminism are better understood as cases of 'missing physics', in that 'standard presentations invariably weasel quite a bit with respect to foundational assumptions that must be settled before a feature such as determinism can be coherently adjudicated' (Wilson [2009], pg. 181). What kind of foundational assumptions? In the case analysed in this paper, the foundational assumptions concern such basic matters as whether classical mechanics should have, in its domain, any possible spatial configuration of infinitely many particles of any possible mass. The answer to how we fill in the 'missing physics' of admissible states in the domain of classical mechanics will decide whether classical mechanics is incomplete or not.

[14]I am grateful to an anonymous referee for pointing out that the unmediated action-at-a-distance of Newtonian gravitational theory, which gives rise to the problems discussed in this paper, might itself be viewed as nothing more than a mathematically convenient modelling assumption. Indeed, Seeliger himself proposed modifying Newton's law of gravity by adding an attenuation factor of $e^{-\lambda r}$, for some $\lambda > 0$, which ensured that the gravitational force converged (see Norton [1999], pg. 294). And alternative forms of the gravitational potential were explored by Neumann ([1877]). In modern particle physics, alternative forms of the gravitational potential are explored in order to prevent these kinds of issues from arising. The important point is that

scribe and model the physical world because that is one of the few tools we have which is adequate for the job. The problem we face is that *sometimes* mathematical reification leads to genuine discoveries, as when Dirac took seriously the negative energy solution to his relativistic version of the Schrödinger wave equation, and in so doing paved the way for the discovery of antimatter. But sometimes mathematical reification creates pseudoproblems.

It is, perhaps, fitting that the demonstration of the incompleteness of classical mechanics used in this paper relied on the same mathematical fact used by the Pasadena game to generate trouble for standard decision theory. Towards the end of their paper, when they consider possible responses, Nover and Hájek reject the suggestion that the Pasadena game be ignored (and, therefore, not seen to present a real problem for decision theory) because the game involves an infinite state space. One part of their reasoning runs as follows:

> The first response is to balk at the game's infinite state space: the assumption that the coin could land heads for the first time on the first toss, or the second, or the third, *ad infinitum*. More generally, the response is that decision theory should be confined to actions that have finitely many consequences—that is, to decision problems that have finitely many states. What could motivate this response?
>
> On the one hand, it might be purely theoretical considerations: it might be claimed that *as a matter of conceptual necessity*, all decision problems have finitely many states. But then the response strikes us as high-handed: for we have no trouble countenancing infinitely many states elsewhere in our theorizing—in physics, for example. (Nover and Hájek [2004], pg. 246)

Yet what we have seen is that one must be careful in countenancing infinite state spaces *even in physics* (at least Newtonian physics), because a problem structurally analogous to that posed by the Pasadena game can arise there, as well. And, in fact, there will be many mathematical models of systems for which the same trick can be used to generate apparent paradoxes. As long as the model needs to aggregate infinitely many values of the right magnitude, both positive and negative, and there is no natural order to aggregate them, an incompleteness result will probably exist.[15]

such modifications often make virtually no practical difference, from an empirical perspective, but have huge conceptual implications. For a discussion of some recent approaches adopted in contemporary physics, see Pitts ([2011]).

[15]Consider a democracy containing infinitely many people where they use majority rule to make decisions. Majority rule is defined as follows: a vote *for* a proposal is represented as $+1$ and a vote *against* a proposal is represented by -1, and the proposal passes if the sum of all votes is greater

Since mathematical anomalies can easily arise when we model systems, we need to distinguish between anomalies which depend purely upon assumptions introduced for reasons of mathematical convenience, and those which do not. This distinction matters because, I suggest, only puzzling results of the second category present interesting challenges for how we interpret and understand a theory. I would put Dirac's interpretation of the negative energy solution in the second category, but the argument for the incompleteness of classical mechanics discussed here — as well as the problem of how to value the Pasadena game — in the first category. The challenge, of course, when faced with a puzzling result that turns on a mathematical assumption, is deciding to which category the assumption belongs.

Acknowledgements

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A The general form of the calculations.

Let us first establish some notational conventions. The *target mass* will be one of the masses m_k as set out in the initial construction from section 2. This mass generates a force F_k on m_0, which is the *target force*. Generating a gravitationally equivalent decomposition involves showing, for each m_k, how pairs of objects, each having the same *source mass* m_s, can be positioned so that the overall set generates a gravitational force equivalent to F_k. In what follows, we shall first derive formulae for the specific case where $m_s = \frac{1}{G}$, and then consider some generalisations. Although the math might look a bit ugly, setting up the equations involves literally nothing more than high school trigonometry.

To begin, note that, given some target mass m_k, each of the pairs of masses used to generate the gravitationally equivalent decomposition for F_k will be located at slightly different places. Let us introduce some notation so that we may

than 0. Suppose that each voter is assigned an unique ID number such that for each $k \in \mathbb{N}$ there is exactly one person with that ID number. Now suppose an election is held to decide Proposition P. Suppose everyone with an odd ID number votes for P and everyone with an even ID number votes against P. Does Proposition P pass?

refer to these pairs of objects with precision. For the target mass m_k, we will use $m^+_{k,1}$ and $m^-_{k,1}$ to refer to the members of the *first* pair of objects. This notation is chosen because the x-coordinate of each object is the same, and their y-coordinates have the same absolute value, symmetrically reflected along the x-axis. The *second* pair of objects will be labelled $m^+_{k,2}$ and $m^-_{k,2}$, and so on. (In cases where we do not need to differentiate between the two members of a pair, we shall omit the superscript.) The force exerted by an individual $m_{k,i}$ on m_0 will be denoted $F_{m_{k,i}}$ and its x and y components will be denoted by $F^x_{m_{k,i}}$ and $F^y_{m_{k,i}}$ respectively. The angle of $F_{m_{k,i}}$, measured from the x-axis, will be denoted by $\theta_{m_{k,i}}$. To construct the gravitational force equivalent of F_k, when $m_s = \frac{1}{G}$ we need to introduce k pairs of objects such that each *pair* exerts a horizontal force of $\frac{1}{k^2}$ on m_0.

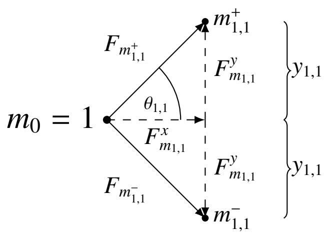


Figure 7: The gravitationally equivalent decomposition for the first target mass m_1.

Let us begin by determining the gravitationally equivalent decomposition for the target mass m_1, as shown in figure 7. The mass $m^+_{1,1}$ exerts a force of $F^+_{m_{1,1}}$ on m_0 and the mass $m^-_{1,1}$ exerts a force of $F^-_{m_{1,1}}$ on m_0. Notice that, due to the symmetrical positioning of the pair, the y-components of $F^+_{m_{1,1}}$ and $F^-_{m_{1,1}}$ cancel out. Since the target force is $F_1 = 1$, we need $2F^x_{m_{1,1}} = 1$. Since $F^x_{m_{1,1}} = F_{m_{1,1}} \cos\theta_{1,1}$, we can now determine the required positions of $m^+_{1,1}$ and $m^-_{1,1}$.

Since each $m_{1,1}$ is located at $x = 1$ with a vertical displacement of $y_{1,1}$, each $m_{1,1}$ is at a distance $r = \sqrt{1^2 + y^2_{1,1}}$ from m_0. And so we know that

$$F_{m_{1,1}} = G\frac{m_{1,1}m_0}{r^2} = G\frac{\frac{1}{G}\cdot 1}{1^2 + y^2_{1,1}} = \frac{1}{1^2 + y^2_{1,1}}$$

and

$$\cos\theta_{1,1} = \frac{1}{\sqrt{1^2 + y^2_{1,1}}}.$$

Solving for $2F^x_{m_{1,1}} = 2F_{m_{1,1}} \cos\theta_{1,1} = 1$ is thus equivalent to solving

$$\frac{1}{\left(1^2 + y_{1,1}^2\right)^{3/2}} = \frac{1}{2}.$$

An approximate numerical solution has the pair $m_{1,1}$ located at $(1, -0.7664)$ and $(1, 0.7664)$.

Now consider the gravitationally equivalent decomposition for the target mass m_2, with target force $F_2 = \frac{1}{2}$. This will be constructed using two pairs of objects, $m_{2,1}$ and $m_{2,2}$, each pair exerting an aggregate horizontal force of $\frac{1}{4}$ on m_0. For the first pair, as shown in figure 8(a), we want to find a vertical displacement $y_{2,1}$ for $m^+_{2,1}$ and $m^-_{2,1}$ such that $2F^x_{m_{2,1}} = \frac{1}{4}$. As before,

$$F^x_{m_{2,1}} = F_{m_{2,1}} \cos\theta_{m_{2,1}}$$

and we know that the pair of objects $m_{2,1}$ is located at $x = -2$, so

$$\begin{aligned} F_{m_{2,1}} &= G\frac{m_{2,1}m_0}{2^2 + y_{2,1}^2} \\ &= G\frac{\frac{1}{G}\cdot 1}{2^2 + y_{2,1}^2} = \frac{1}{2^2 + y_{2,1}^2}. \end{aligned}$$

Now, $\cos\theta_{2,1} = \frac{2}{\sqrt{2^2+y_{2,1}^2}}$, and so

$$F^x_{m_{2,1}} = \frac{2}{\left(2^2 + y_{2,1}^2\right)^{3/2}}.$$

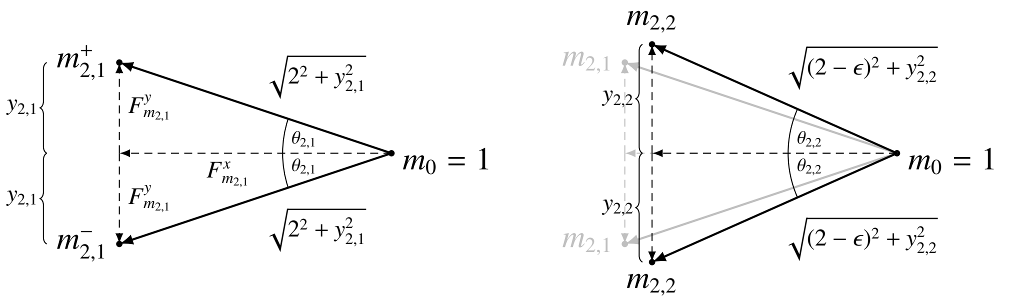
Solving the equation $2F^x_{m_{2,1}} = \frac{1}{4}$ for $y_{2,1}$ yields $y_{2,1} = \pm 2\sqrt{-1 + 2^{2/3}}$. A numerical approximation is $y_{2,1} = \pm 1.53284$.

To find the location of the second pair $m_{2,2}$, we offset the x-position of this pair by some amount ϵ towards the origin. Thus we have

$$F_{m_{2,2}} = G\frac{m_{2,2}m_0}{(2-\epsilon)^2 + y_{2,2}^2} = \frac{1}{(2-\epsilon)^2 + y_{2,2}^2}.$$

Since $\cos\theta_{2,2} = \frac{2-\epsilon}{\sqrt{(2-\epsilon)^2+y_{2,2}^2}}$, it follows that

$$F^x_{m_{2,2}} = \frac{2-\epsilon}{\left[(2-\epsilon)^2 + y_{2,2}^2\right]^{3/2}}.$$



(a) Step 1. The masses $m_{2,1}$ are positioned at $x = -2$ at some initially unknown vertical displacement $y_{2,1}$.

(b) Step 2. The masses $m_{2,2}$ are positioned at $x = -2 + \epsilon$ at some unknown vertical displacement $y_{2,2}$.

Figure 8: A diagram showing the relevant calculations for constructing the gravitationally equivalent decomposition for the target mass m_2.

If we assume that $\epsilon = \frac{1}{10}$, a numerical solution to the equation $2F^x_{m_{2,2}} = \frac{1}{4}$ is $y_{2,2} = \pm 1.58939$. This makes sense, for moving the pair of objects closer to m_0 without increasing the vertical displacement would generate a horizontal force greater than $\frac{1}{4}$, and so the vertical displacement has to be increased as an adjustment.

Now consider the target mass m_3, located at $x = 3$. Going through a similar process of reasoning as before, we find that $F_{m_{3,1}} = \frac{1}{3^2+y_{3,1}^2}$, $\cos\theta_{3,1} = \frac{3}{\sqrt{3^2+y_{3,1}^2}}$, and so $F^x_{m_{3,1}} = \frac{3}{\left(3^2+y_{3,1}^2\right)^{3/2}}$. We solve for $y_{3,1}$ which satisfies the equation $2F^x_{m_{3,1}} = \frac{1}{9}$. (This is because the gravitationally equivalent decomposition will be constructed with three pairs of objects, each pair of objects exerting a force of $\frac{1}{9}$ on m_0.) Solving the equation numerically yields $y_{3,1} \approx \pm 2.29926$. The equations for the values of $y_{3,2}$ and $y_{3,3}$ take the form:

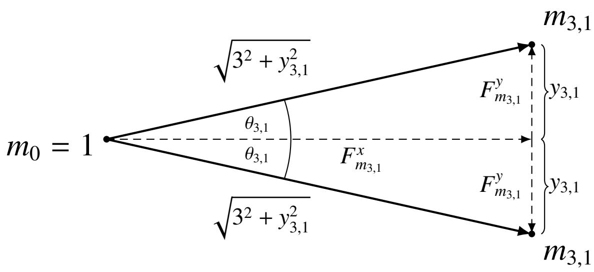
$$2F^x_{m_{3,2}} = 2 \cdot \frac{3-\epsilon}{\left[(3-\epsilon)^2 + y_{3,2}^2\right]^{3/2}} = \frac{1}{9}$$

and

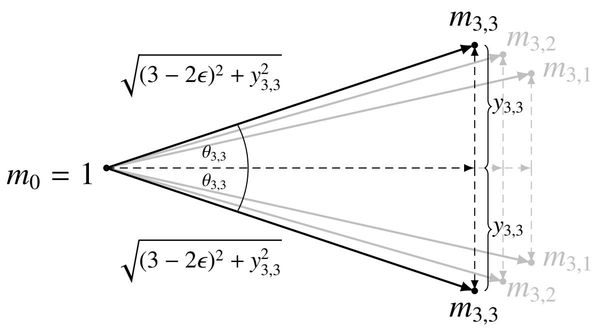
$$2F^x_{m_{3,3}} = 2 \cdot \frac{3-2\epsilon}{\left[(3-2\epsilon)^2 + y_{3,2}^2\right]^{3/2}} = \frac{1}{9}$$

with approximate solutions $y_{3,2} \approx \pm 2.3574$ and $y_{3,3} \approx \pm 2.40923$, respectively.

In general, given object m_N with mass $\frac{N}{G}$ from the original construction of section 2, we can find N pairs of objects, each with mass $\frac{1}{G}$, which collectively



(a) Step 1. The masses $m_{3,1}$ are positioned at $x = 3$ at some initially unknown vertical displacement $y_{3,1}$.



(b) Step 2. The masses $m_{3,3}$ are positioned at $x = 3 - 2\epsilon$ at some unknown vertical displacement $y_{3,3}$.

Figure 9: A diagram showing some of the relevant calculations for constructing the gravitationally equivalent decomposition for the target mass m_3.

exert a gravitational force equivalent to F_N on m_0. The y-positions of the $m_{N,k}$, for $k = 1, \ldots, N$, are obtained by solving for $y_{N,k}$ in an equation of the following generic form:

$$2F^x_{m_{N,k}} = 2 \cdot \frac{N - (k-1)\epsilon}{\left[(N - (k-1)\epsilon)^2 + y^2_{N,k}\right]^{3/2}} = \frac{1}{N^2}. \tag{2}$$

What changes if we construct the gravitationally equivalent decomposition with a smaller source mass for each pair of objects? Suppose we were to use $m_s = \frac{1}{10^n G}$, for some integer n, with each pair of objects exerting an external force of $\frac{1}{10^n}$ on m_0. This will require using 10^n pairs of objects, and will require selecting an appropriately small value of ϵ (taking $\epsilon = \frac{1}{10^n}$ will work). In this case, equation (2) takes the following form:

$$2F^x_{m_{N,k}} = 2 \cdot \frac{\frac{1}{10^n}(N - (k-1)\epsilon)}{\left[(N - (k-1)\epsilon)^2 + y^2_{N,k}\right]^{3/2}} = \frac{1}{10^n \cdot N^2}. \tag{3}$$

When $n = 1$ and $\epsilon = \frac{1}{10}$, this yields the gravitationally equivalent decomposition shown in figure 6(a).

One further change can be made in order to generate a gravitationally equivalent decomposition that approximates continuous curves: suppose that we pick a source mass of $m_s = \frac{1}{10^n G}$ but instead of positioning pairs of objects, each having mass m_s, we position pairs of objects each with mass $\frac{1}{2}m_s + \delta$ for some very small $\delta > 0$. When δ is very small, this means the vertical displacement around the x-axis for $m^+_{N,1}$ and $m^-_{N,1}$ will be very tiny. In this case, equation (2) takes the form:

$$2F^x_{m_{N,k}} = 2 \cdot \frac{\left[\frac{1}{2} \cdot \frac{1}{10^n} + \delta\right](N - (k-1)\epsilon)}{\left[(N - (k-1)\epsilon)^2 + y^2_{N,k}\right]^{3/2}} = \frac{1}{10^n \cdot N^2}. \tag{4}$$

Setting $n = 2$, $\epsilon = \frac{1}{100}$, and $\delta = 10^{-6}$ yields the gravitationally equivalent decomposition shown in figure 6(b).

B Approximating Seeliger's paradox in a discrete limit

Here we show how a suitable modification of the original construction yields, in the limit, ever-greater discrete approximations of a uniform mass distribution of the kind underlying Seeliger's paradox. To do this, we will construct a sequence of discrete incompleteness results $\langle D_n \rangle_{n=0}^{\infty}$, and show that the distribution of particles in two dimensions is such that, for an open ball B of some radius r, the amount of mass contained in B, denoted $m(B)$, converges to the measure of B, denoted $\mu(B)$, as $n \to \infty$.

First, notice that for any positive integer n that if we place a unit mass m_0 at the origin and objects of mass $m_k = \frac{k}{10^n \cdot G}$ at the points $x_k = (-1)^{k-1} \cdot \frac{k}{10^n}$ we obtain a scaled version of the original problem. The force contribution $F_k^{(n)}$ of the k^{th} object is

$$F_k^{(n)} = G \frac{m_0 m_k}{\left(\frac{k}{10^n}\right)^2} = G \frac{1 \cdot \frac{k}{10^n G}}{\frac{k^2}{10^{2n}}} = \frac{10^n}{k}.$$

And so when we calculate the total force acting on the mass m_0 we get the sum

$$F^{(n)} = \sum_{k=1}^{\infty} = \frac{10^n}{1} - \frac{10^n}{2} + \frac{10^n}{3} - \frac{10^n}{4} + \frac{10^n}{5} - \frac{10^n}{6} + \cdots$$

which is the conditionally convergent series obtained by multiplying the alternating harmonic series by a factor of 10^n, and so the incompleteness result obtains. The difference between the original incompleteness result and this version is that the space between the objects is scaled by a factor of 10^{-n}.

Second, notice that for any rescaled version of the incompleteness result it remains possible to construct a gravitationally equivalent decomposition, using (i) a sufficiently small source mass, say $m_s = \frac{1}{10^{2n} G}$, (ii) an appropriately chosen value of ϵ (we could take $\epsilon = \frac{1}{10^{2n}}$), and (iii) pairs of objects with mass $\frac{1}{2} \cdot m_s + \delta$, as discussed at the end of appendix A, such that the arcs formed will increasingly approximate a continuous curve, as shown in figure 6(b). Importantly, these discrete curve approximations will become arbitrarily close together as n increases.

Third, notice that, as shown in figure 10, for each mass m_k we can find a compact set S_k which contains all of the point masses in the gravitationally equivalent decomposition of m_k with the property that the area of S_k, denoted $\mu(S_k)$, is proportional to m_k. In particular, it is possible to choose all of the S_k such that there exists some common constant $c > 0$ such that $\mu(S_k) = c \cdot m_k$, for all k. Moreover, we can choose the S_k such that all of the S_k are pairwise disjoint. (This is because the amount of mass that appears at location x_k only increases linearly as we move from x_k to x_{k+1} but area increases as the square of the distance.) Since the incompleteness result does not change if we rescale of the masses m_k by a common positive constant, we can assume, without loss of generality, that $\mu(S_k) = m_k$, for all k.

Given these three observations, we construct a sequence of discrete incompleteness results which increasingly approximate a uniform mass distribution, in the limit, as follows. Let $\langle D_n \rangle_{n=0}^{\infty}$ be the sequence of mass distributions defined as follows (superscripts appearing in parentheses are indexes, not powers):

1. The position of the target masses in D_n is given by $x_k^{(n)} = (-1)^{k-1} \cdot \frac{k}{10^n}$,

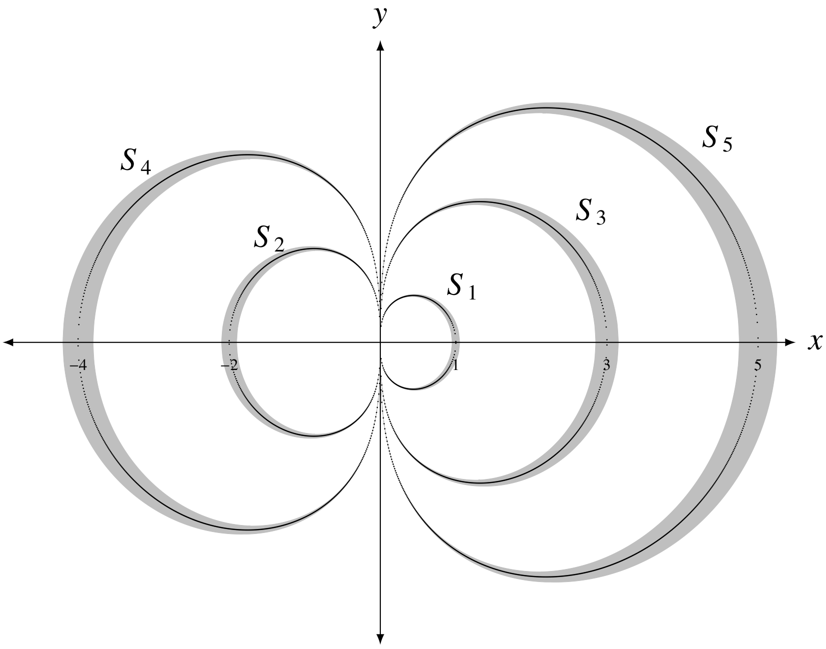


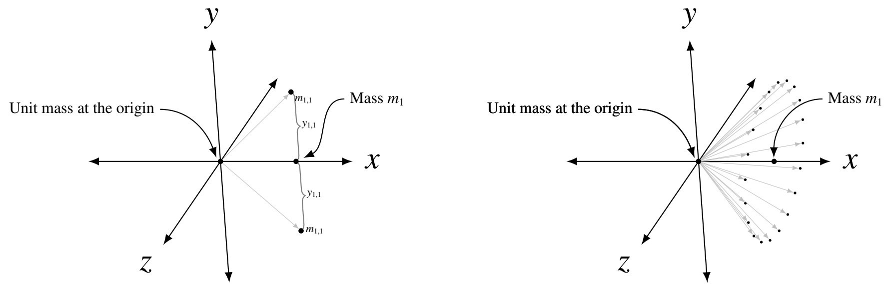
Figure 10: A family of disjoint closed sets,

2. The target mass $m_k^{(n)} = \frac{k}{c_n 10^n G}$, where the constant $c_n > 0$ is chosen to ensure that $\mu\left(S_k^{(n)}\right) = m_k$,
3. The gravitationally equivalent decomposition of the target mass in D_n is performed with a source mass of $m_s^{(n)} = \frac{1}{10^{2n} G}$ and pairs of objects having a mass of $\frac{1}{2} \cdot m_s^{(n)} + \delta_n$, where $\delta_n \to 0$ as $n \to \infty$.

Let $B \subseteq \mathbb{R}^2$ be an open ball of some radius r. For any particular n, each of the $S_k^{(n)}$ is measurable and hence $\bigcup_k \left(B \cap S_k^{(n)}\right)$ is measurable. Since the crescents $S_k^{(n)}$ get arbitrarily close together as $n \to \infty$, the measure of $\bigcup_k \left(B \cap S_k^{(n)}\right)$ will approach $\mu(B)$ in the limit. Since, for all n and k, we have $\mu\left(S_k^{(n)}\right) = m\left(S_k^{(n)}\right)$, the amount of mass contained in the open ball B will approach the measure of B. Since B was an arbitrary open ball, and measure is preserved under translation, the discrete distribution of mass increasingly approximates a uniform mass distribution as $n \to \infty$.

C From two to three dimensions

The construction given only concerns the positioning of particles within a two-dimensional space. Since Seeliger's original paper concerned a uniform mass



(a) Constructing a simple gravitationally equivalent decomposition in the xz-plane with two objects each having mass $m_{1,1}$ equal to m_1.

(b) Constructing a more complex gravitationally equivalent decomposition in three dimensions using ten pairs of objects, each object with mass equal to $\frac{1}{10G}$.

Figure 11: Performing a gravitationally equivalent decomposition for m_1 in three dimensions.

distribution in three-dimensional space, it is worth showing how the construction can be extended to three dimensions. Once it is seen how to do this, it will be clear how to modify the proof given in Appendix B.

To begin, consider performing a gravitationally equivalent decomposition of $m_1 = \frac{1}{G}$ using *two* objects of the same mass. (In the notation established earlier, this means that $m_{1,1} = m_1 = \frac{1}{G}$.) Figure 11(a) shows what the arrangement of masses would look like if they were positioned at $x = 1$ but shifted the appropriate amount $\pm y_{1,1}$ along the y-axis.

Now, though, suppose that instead of taking $m_{1,1}$ to equal m_1 we instead choose $m_{1,1} = \frac{1}{10G}$. In this case, we would have $F_{m_{1,1}} = \frac{1}{20}$ and so a single pair of masses as in figure 11(a) would only contribute one-tenth of the original force of m_1 on m_0. However, adding ten pairs of masses, *rotated around* the x-axis at the same distance d, as shown in figure 11(b), would result in a net gravitational force equal to that exerted by m_1 on m_0. If we instead chose $m_{1,1} = \frac{1}{100G}$, we would need to use one hundred pairs of masses positioned in a ring around the x-axis. As $m_{1,1} \to 0$, we get ever-closer discrete approximations to a continuous distribution of mass along a ring, with radius $y_{1,1}$, centered at $x = 1$.

This construction can easily be extended to handle the other cases discussed earlier in this paper. Whereas previously we only considered positioning a pair of objects, each with mass $m_{n,k}$, at symmetric positions above and below the x-axis, now we also utilise the rotational symmetry around the x-axis to position additional pairs of objects, spreading into three dimensions. The only additional change when we extend the construction into three dimensions is that the value of the mass $m_{n,k}$ needs to be reduced because we are introducing additional pairs of

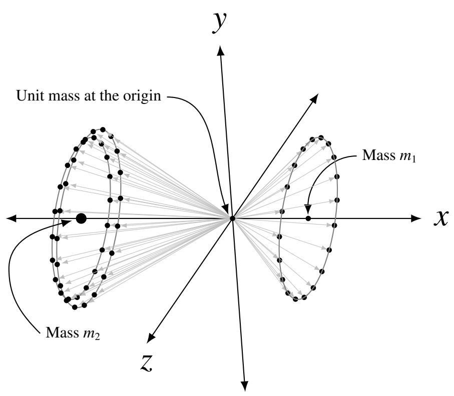


Figure 12: The construction of a gravitationally equivalent decomposition, in three dimensions, for the mass m_2. Here, we begin with two pairs of masses positioned in the xy-plane, as shown in figure 3, but then position nine additional pairs of masses rotated around the x-axis. Note that in order to make each pair of masses visible in the diagram, they are not drawn to scale (recall that each mass in a pair has the value $\frac{1}{10G}$). A ring is drawn in order to make the rotational symmetry around the x-axis more clear.

objects rotated around the x-axis, along the lines illustrated in figure 11. Figure 12 shows the gravitationally equivalent decompositions for both m_1 and m_2.

Given this, the proof given in the previous section needs to be modified so as to have each of the S_k be three-dimensional 'shells' containing the gravitationally equivalent decomposition, rather than two-dimensional strips. It also needs to be the case that the constant c_n is chosen so as to ensure that the *volume* of each of the S_k equals the target mass $m_k^{(n)}$. Otherwise the essential details remain the same.

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